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Algorithms & Data Structures Homework 0 HS 18

Exercise 0.1 *Find the Shortest Path.*

Consider a hypothetical floor plan such that the area is organized in hexagonal cells as shown in Figure 1. We begin at the cell marked *start*, and we want to reach the cell marked *end*. We can travel from one cell to another, if the two cells are neighbouring, i.e. they share an edge. Each time we cross from one cell to a destination cell, we consider the move as a single step and the destination cell as visited. We want to find the shortest path from the start to the end, minimizing the number of steps.

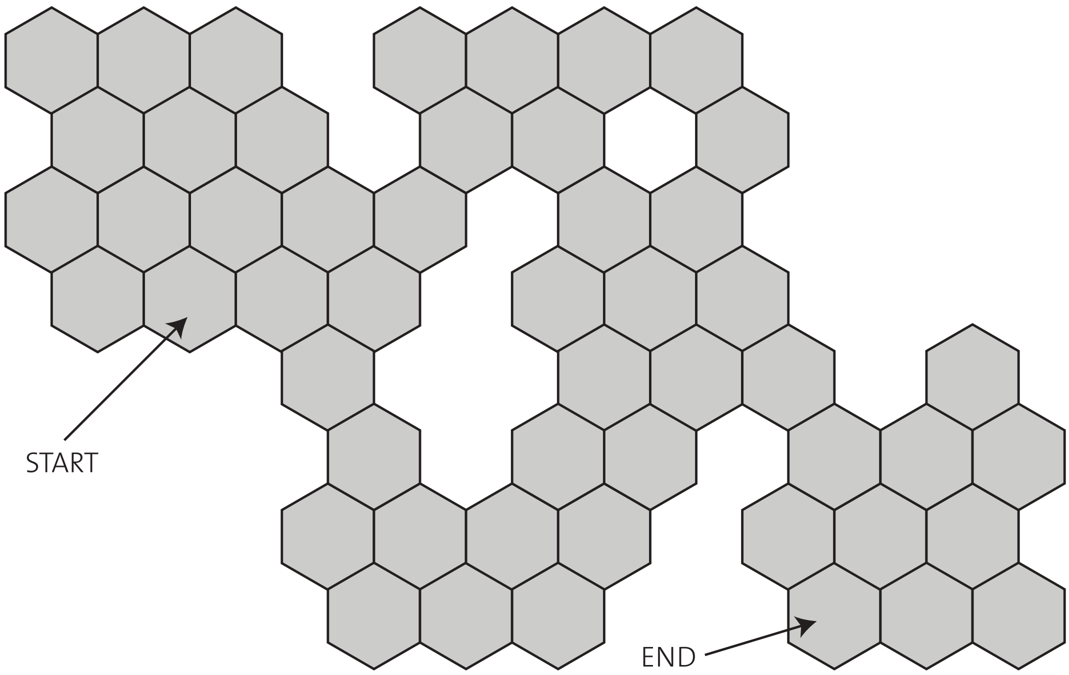

Figure 1: Floor plan

Consider the following algorithm:

1. We consider the start cell as visited, and we write the number zero on the cell.
2. If a cell has the number n written on it, we can visit all neighbouring cells, writing the number $n+1$ on it if either:
   - the neighbouring cell has not been visited before, or
   - the neighbouring cell has been visited before and holds the number m such that $m > n+1$.
3. We stop the execution of the algorithm when we can no longer execute 2.

Your tasks:

1. Execute the algorithm on the floor plan given in Figure 1, writing numbers on each cell.

   Solution:

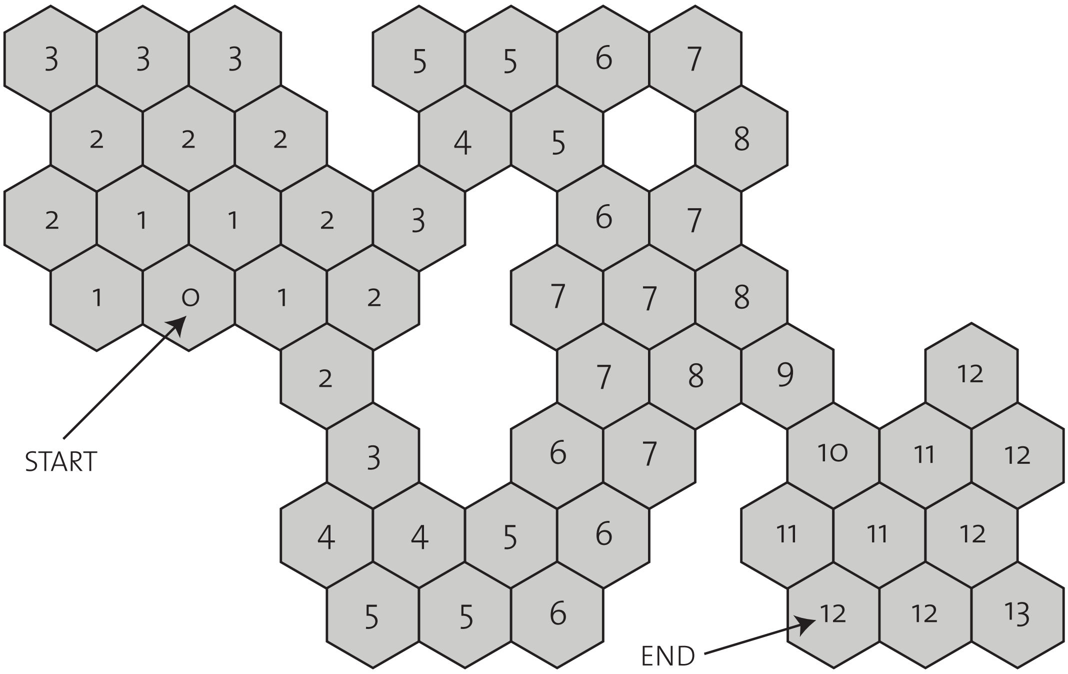

Figure 2: The state of the floor plan, once the algorithm is done with execution

2. How many steps do we need to reach the end cell?

   Solution: Once the algorithm stops executing, each cell will contain a number that corresponds to the shortest distance from the start cell. As a result, we will reach the end cell in 12 steps.

3. For a given plan with N cells including the start and the end cell, how many steps do we need to reach the end cell, starting from the start cell, in the worst case scenario?

   Solution: The worst case scenario is having a floor plan in which each cell has only two neighbouring cells, such that the two neighbouring cells do not share an edge, and the start and the end cell have only one neighbouring cell. In this case scenario, we would need $N - 1$ steps to reach the end cell starting from the start cell.

Exercise 0.2 *Induction.*

1. Prove via mathematical induction, that the following holds for any positive integer n:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{(n)(n+1)(2n+1)}{6}$$

   - **Base Case.**
     Let $n = 1$. Then:

$$1 = \frac{(1)(1+1)(2+1)}{6} = 1$$

   - **Induction Hypothesis.**

Assume that the property holds for some postiive integer k. That is:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{(k)(k+1)(2k+1)}{6}$$

- **Inductive Step.**

  We must show that the property holds for $k+1$. Add $(k+1)^2$ to both sides of our inductive hypothesis.

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \frac{(k)(k+1)(2k+1)}{6}+(k+1)^2 = \frac{(k)(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+k+6k+6)}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

  By the principle of mathematical induction, this is true for any positive integer n.

2. Prove via mathematical induction that for any positive integer n, $2n^3 + 3n^2 + n$ is divisible by 6.

   - **Base Case.**
     Let $n = 1$. Then $2(1^3) + 3(1^2) + 1 = 6$
   - **Induction Hypothesis.**
     Assume that the property holds for some positive integer k. That is, there exists some positive integer m such that:
     $$6m = 2k^3 + 3k^2 + k$$
   - **Inductive Step.**
     We must show that the property holds for $k+1$.

     $$2(k+1)^3 + 3(k+1)^2 + k + 1 = 2k^3 + 6k^2 + 6k + 2 + 3k^2 + 6k + 3 + k + 1$$
     $$= 2k^3 + 3k^2 + k + 6k^2 + 12k + 6$$

     From our induction hypothesis, $2k^3 + 3k^2 + k = 6m$.

     $$2(k+1)^3 + 3(k+1)^2 + (k+1) = 6m + 6(k^2 + 2k + 1)$$

     And this is divisible by 6.

     By the principle of mathematical induction, this is true for any positive integer n.

Exercise 0.3 *Coloring a Map.*

Given is a map that is divided by n (pairwise different) straight lines. You want to color the regions on the map (i.e., the areas bordered by the lines), such that no two neighbouring regions (i.e., regions that share a common segment of a line as a border) get the same color.

Prove by mathematical induction on n, that you can color every such map with 2 colors.

- **Base Case.**
  Let $n = 1$. In this case, there are exactly two regions. We color one black and the other one white.

- **Induction Hypothesis.**
  Suppose that a map defined by k lines can be colored with two colors, for some positive integer k.
- **Inductive Step.**
  We must show that the property holds for $k+1$. Consider any of the $(k+1)$ lines: the line divides the map into a left part and a right part, both containing at most k lines. By induction hypothesis, we can thus color both parts using two colors. Now either this gives already a correct coloring for the whole map, or, flipping the colors in the right part does.

Homework: This homework does not have to be submitted and will be discussed in the first exercise session on 24.9.2018